Shape and Topology Optimization Methods for Inverse Problems

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joint works with:
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Der Wissenschaftsfonds.

START PROJECT
INTERFACES/FREE BOUNDARIES

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Inverse Problems

- Direct problem: given initial conditions
  $\longrightarrow$ find evolution of a physical system
- Inverse problem: given final state of a physical system
  $\longrightarrow$ find initial conditions
- Well-posedness (Hadamard):
  1. Existence
  2. Uniqueness
  3. Stability
- The problem is Ill-posed if one of these conditions is not satisfied $\longrightarrow$ need for regularization.
- Many important inverse problems in tomography, geology, etc ... are ill-posed.

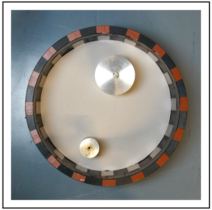
Electrical Impedance Tomography (EIT)

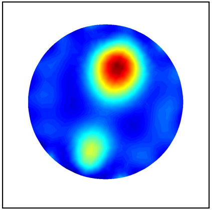
Ω bounded, simply connected, $\Sigma = \partial\Omega$.

- Given: apply electric currents f on Σ / measure voltages v on Σ.
- Find: electrical properties in Ω matching measurements.

At $x \in \Omega$, find admittivity $\gamma(x,\omega) = q(x) + i\omega e(x)$ with

- electrical conductivity q;
- electric permittivity e





courtesy: Dept. Physics, Univ. Kuopio

EIT – Potential applications

In many applications, there is a large resistivity ($1/q$) contrast between a wide range of materials (e.g., up to about 200:1 for tissue types in the body; Geddes and Baker, 1967).

$\Longrightarrow$ used for imaging structure within Ω.

- Geophysics: Porosity of core samples, map groundwater in borehole-to-borehole experiments, ...
- Medical imaging: Pulmonary measurements (functionality, detection of emboli), breast cancer detection, blood flow,...
- Non-destructive testing: Crack identification, void detection...
- References: [Ammari et al], [Borcea, Guevara Vasquez], [Calderón],[Cheney, Isaacson, Newell], [Druskin et al.], [Hansen,Knudsen], [Kaipio et al.], [Kohn, Vogelius], [Lionheart], [Nachman], [Somersalo et al.], [Sylvester, Uhlmann]...

EIT – Mathematical model

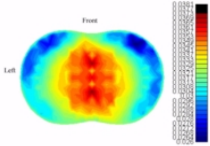
The model comes in two flavors:

- Continuum model. f supposed to be known on all of Σ.
- Electrode model. Current (or voltage) through electrodes distributed along Σ.

In practice, boundary currents $f(x)$ are not known for all $x \in \Sigma$.

- Currents sent along wires attached to N electrodes.

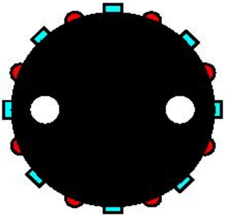


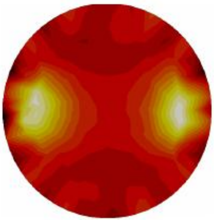


courtesy: EIT group, Oxford Brookes Univ.

Fluorescence Optical Tomography (FOT)

1. Diffusion of photons at e(x)citation wavelength λ_x from sources at the boundary into the body.
2. Absorption at λ_x by fluorophores and re-e(m)ission at wavelength λ_m .
3. Diffusion of re-emitted photons through the body.
4. Measurement of light intensities leaving the body.
5. Find c, the concentration of fluorophores in Ω.





courtesy: Manuel Freiberger, Univ. Graz

FOT – Potential applications

- FOT uses fluorescent dyes to overcome the low contrast in optical parameters, that result in low signal-to-noise ratios.
- Medical imaging.
- Low-cost alternative or complement to existing imaging technology such as EIT.
- References: [Arridge], [Chance et al.], [Dorn et al.], [Egger et al.], [Roy et all], [Schweiger et al.] ...

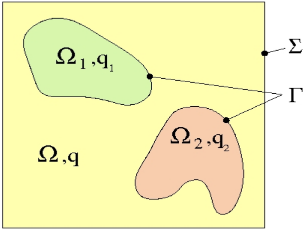
Piecewise constant model

- We are interested in sharp interface models and reconstructions [Chan et al.], [Dorn et al.], [Eckel, Kress]....
- Piecewise constant conductivity

$$q(x) = \sum_{i=0}^{n_q} q_i \chi_{\Omega_i}(x)$$

.

- Unknowns: q_i and interface $\Gamma := \bigcup_{i=1}^{n_q} \Gamma_i$, $\Gamma_i := \partial\Omega_i$.



EIT – Output-least-squares formulation

Given $f_k(x) \in H^{-1/2}(\Omega)$ and associated measurements $m_k \in L^2(\Sigma)$

$$\begin{aligned} \text{minimize} \quad & J(q) = \frac{1}{2}\sum_{k=1}^{M} \|u_k(q) - m_k\|^2_{L^2(\Sigma)} + \alpha R(q) \\ \text{subject to} \quad & \begin{aligned} \operatorname{div}(q\nabla u_k) &= 0 \quad && \text{in } H^1(\Omega)', \\ q\partial_n u_k &= f_k && \text{on } \Sigma, \\ \int_\Sigma u_k &= 0, && k = 1, \dots, M. \end{aligned} \end{aligned}$$

Regularization: Total variation (TV)

$$R(q) = \int_\Omega |Dq| = \sum_{i=1}^{n_q} |q_0 - q_i| \operatorname{Per}(\Omega_i).$$

EIT – Output-least-squares formulation

Given $f_k(x) \in H^{-1/2}(\Omega)$ and associated measurements $m_k \in L^2(\Sigma)$

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\end{aligned}
$$

Shape optimization perspective

$$
\mathcal{J}(\{\Omega_i, q_i\}_{i=0}^{n_q}) = J\left(\sum_{i=0}^{n_q} q_i \chi_{\Omega_i}(x)\right)
$$

FOT – Output-least-squares formulation

Given $f_k(x) \in (H^1(\Omega))'$ and associated measurements $m_k \in L^2(\Sigma)$

$$\begin{array}{llll}
\text{minimize} & J(c) = \frac{1}{2}\sum_{k=1}^{M} \|\rho_{\mathrm{m}}\phi_{\mathrm{m},k}(c) - m_k\|^2_{L^2(\Sigma)} + \alpha R(c) & & \\
\text{subject to} & \mathrm{div}(\kappa_{\mathrm{x}}(c)\nabla\phi_{\mathrm{x},k}) + \mu_x(c)\phi_{\mathrm{x},k} = f_k & & \text{in } H^1(\Omega)', \\
& \kappa_{\mathrm{x}}(c)\partial_n\phi_{\mathrm{x},k} + \rho_{\mathrm{x}}\phi_{\mathrm{x},k} = 0 & & \text{on } \Sigma, \\
& \mathrm{div}(\kappa_{\mathrm{m}}(c)\nabla\phi_{\mathrm{m},k}) + \mu_m(c)\phi_{\mathrm{m},k} = \gamma(c)\phi_{\mathrm{x},k} & & \text{in } H^1(\Omega)', \\
& \kappa_{\mathrm{m}}(c)\partial_n\phi_{\mathrm{m},k} + \rho_{\mathrm{m}}\phi_{\mathrm{m},k} = 0 & & \text{on } \Sigma,
\end{array}$$

Regularization: Total variation (TV)

$$R(c) = \int_\Omega |Dc| = \sum_{i=1}^{n_c} |c_i - c_0|\mathrm{Per}(\Omega_i).$$

FOT – Output-least-squares formulation

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$$
\begin{array}{lrcll}
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\text{subject to} & \nabla\cdot(\kappa_{\mathrm{x}}(c)\nabla\phi_{\mathrm{x},k}) + \mu_x(c)\phi_{\mathrm{x},k} & = & f_k & \text{in } H^1(\Omega)', \\
 & \kappa_{\mathrm{x}}(c)\partial_n\phi_{\mathrm{x},k} + \rho_{\mathrm{x}}\phi_{\mathrm{x},k} & = & 0 & \text{on } \Sigma, \\
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 & \kappa_{\mathrm{m}}(c)\partial_n\phi_{\mathrm{m},k} + \rho_{\mathrm{m}}\phi_{\mathrm{m},k} & = & 0 & \text{on } \Sigma,
\end{array}
$$

Shape optimization perspective

$$
\mathcal{J}(\{\Omega_i, c_i\}_{i=0}^{n_c}) = J(c) = \frac{1}{2}\sum_{k=1}^{M} \|\rho_{\mathrm{m}}\phi_{\mathrm{m},k}(c) - m_k\|^2_{L^2(\Sigma)} + \alpha R(c)
$$

Algorithmic approach

1. Initialize the sub-regions Ω_i using the topological derivative.
2. Update the interface Γ for fixed q_i using the shape derivative and a steepest descent flow within a level set framework.
3. Update the conductivities q_i for fixed Γ.
4. Then alternate step 2 and 3.

Step1: Topological derivative

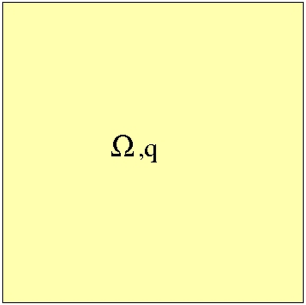
- **Problem of initialization**: Use topological sensitivity.
- Existing works: [Nazarov et al.], [Eschenauer, Schumacher], [Sokolowski, Zochowski], [Masmoudi et al.], [Amstutz], [Ammari et al.], [Bendsøe, Sigmund], [Novotny et al.], ...
- **Topological derivative** of $\mathcal{J}$ at $x \in \Omega$:

$$\mathcal{T}(x) = \lim_{\varepsilon \downarrow 0} \frac{\mathcal{J}(\Omega \setminus B(x;\varepsilon)) - \mathcal{J}(\Omega)}{|B(x;\varepsilon)|}$$

- A small inclusion $B(x;\varepsilon)$ may be created where $\mathcal{T}(x) < 0$

$$\Longrightarrow \mathcal{J}(\Omega \setminus B(x;\varepsilon)) < \mathcal{J}(\Omega)$$

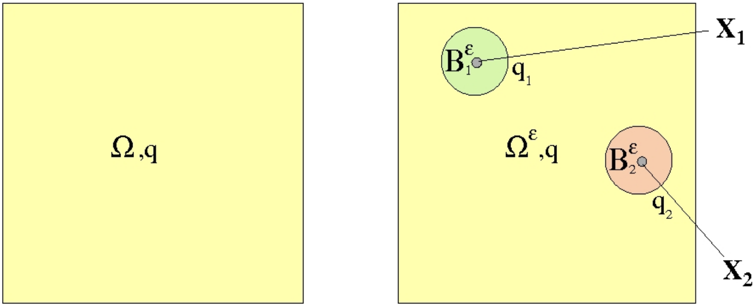
Topological derivative



$$
\begin{aligned}
-\Delta u^0 &= 0 \ \text{ in } \Omega, \\
q\partial_n u^0 &= f \ \text{ on } \Sigma.
\end{aligned}
$$

$$\mathcal{J}(\Omega) = \int_\Sigma (u^0 - m)^2$$

Topological derivative



$$
\begin{aligned}
-\Delta u^0 &= 0 \text{ in } \Omega, \\
q\partial_n u^0 &= f \text{ on } \Sigma.
\end{aligned}
\qquad
\begin{aligned}
-\operatorname{div}(q^\varepsilon \nabla u^\varepsilon) &= 0 \text{ in } \Omega, \\
q^\varepsilon \partial_n u^\varepsilon &= f \text{ on } \Sigma.
\end{aligned}
$$

$$
\mathcal{J}(\Omega) = \int_\Sigma (u^0 - m)^2 \qquad \mathcal{J}(\Omega^\varepsilon) = \int_\Sigma (u^\varepsilon - m)^2
$$

Asymptotic analysis

We perform an asymptotic expansion of the type

$$u^\varepsilon(x) = \sum_{j=0}^{\infty} \varepsilon^j (v_j(x) + W_j(\varepsilon^{-1}x)).$$

- v_j are functions of *regular type*, living in Ω.
- W_j are *boundary layers*, living in $\mathbb{R}^3 \setminus \overline{\omega}$.

Then replace u^ε in $J(\Omega^\varepsilon)$:

$$\mathcal{J}(\Omega^\varepsilon) = \int_\Sigma (u^\varepsilon - m)^2 = \mathcal{J}(\Omega) + |B^\varepsilon| \mathcal{T}_0(x) + \ldots$$

Topological sensitivity

- **First-order expansion** ($n_q = 1$, $\Omega^\varepsilon := \Omega \setminus B^\varepsilon(x)$)

$$\mathcal{J}(\Omega^\varepsilon) = \mathcal{J}(\Omega) + |B^\varepsilon|\mathcal{T}_0(x) \qquad + r_{0,\varepsilon}(x),\ \varepsilon \to 0,$$

- The leading term is

$$\mathcal{T}_0(x) = \frac{N}{N-1}\alpha \nabla u^0(x) \cdot \nabla p(x)$$

- $\alpha = (q - q_1)/(q + q_1/(N-1))$, and p denotes the adjoint state

$$\begin{aligned} -\Delta p &= 0 \text{ in } \Omega, \\ \partial_n p &= 2(u - m) \text{ on } \Sigma. \end{aligned}$$

- $r_{0,\varepsilon}(x) = o(|B^\varepsilon|)$ is the remainder.

Topological sensitivity

- **Second-order expansion** ($n_q = 1$, $\Omega^\varepsilon := \Omega \setminus B(x;\varepsilon)$)

$$\mathcal{J}(\Omega^\varepsilon) = \mathcal{J}(\Omega) + |B^\varepsilon|\mathcal{T}_0(x) + \varepsilon^{2N}\mathcal{T}_1(x) + r_{1,\varepsilon}(x),\ \varepsilon \to 0,$$

- The leading term is

$$\mathcal{T}_0(x) = \frac{N}{N-1}\alpha\nabla u^0(x)\cdot\nabla p(x)$$

- $\alpha = (q - q_1)/(q + q_1/(N-1))$, and p denotes the adjoint state

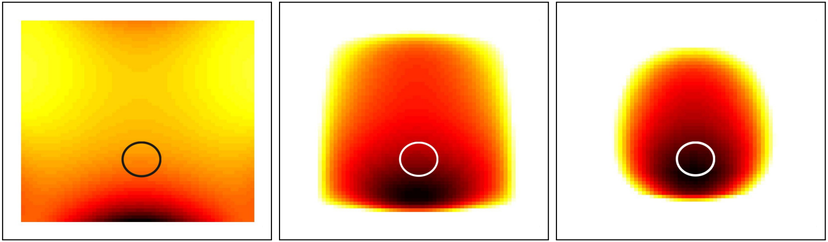
$$\begin{aligned} -\Delta p &= 0 \text{ in } \Omega, \\ \partial_n p &= 2(u - m) \text{ on } \Sigma. \end{aligned}$$

- $r_{1,\varepsilon}(x)$ is the remainder.

Higher-order expansion

$$\mathcal{J}(\Omega^\varepsilon) = \mathcal{J}(\Omega) + |B^\varepsilon|\mathcal{T}_0(x) + \varepsilon^{2N}\mathcal{T}_1(x) + \varepsilon^2|B^\varepsilon|\mathcal{T}_2(x) + r_{2,\varepsilon}(x), \ \varepsilon \to 0,$$

$\mathcal{T}_0(x)$ | $\mathcal{T}_0(x), \mathcal{T}_1(x)$ | $\mathcal{T}_0(x), \mathcal{T}_1(x), \mathcal{T}_2(x)$



$\mathcal{T}_0(x) < 0$ on Σ. Higher-order terms provide a better result.

Higher-order expansion

For one trial inclusion ($n_q = 1$)

$$\mathcal{J}(\Omega^\varepsilon) = \mathcal{J}(\Omega) + |B^\varepsilon|\mathcal{T}_0(x) + \varepsilon^{2N}\mathcal{T}_1(x) + \varepsilon^2|B^\varepsilon|\mathcal{T}_2(x) + O(\varepsilon^{N+3}),$$

with

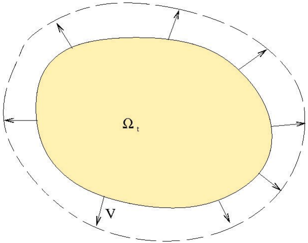
$$\begin{aligned}
\mathcal{T}_0(x) &= \frac{N}{N-1}\alpha\nabla u^0(x)\cdot\nabla p(x),\\
\mathcal{T}_1(x) &= \frac{\alpha^2}{2(N-1)^2}\sum_{i,j}\partial_i u(x)\,\partial_j u(x)\mathcal{I}^{(1)}_{i,j}(x),\\
\mathcal{T}_2(x) &= \frac{\beta}{N}D^2u^0(x)\cdot D^2p(x),
\end{aligned}$$

where

$$\mathcal{I}^{(1)}_{i,j}(x) = \int_\Sigma \frac{(\xi-x)_i(\xi-x)_j}{|\xi-x|^{2N}}d\xi$$

Step 2: Shape optimization

- **General concept:** Smooth boundary transformation.
- [Murat, Simon], [Sokolowski, Zolesio], [Delfour, Zolesio] ...



moving domain Ω_t

Perturbation field: V

Moving domain: $\Omega_t = T_t(V)(\Omega)$

Shape functional: $J(\Omega_t)$

Shape derivative:

$$dJ(\Omega, V) = \lim_{t \to 0} \frac{J(\Omega_t) - J(\Omega)}{t}$$

Step 2: Shape optimization

Consider the domains for n_q inclusions

$$\Omega^* = \Omega \setminus \left(\cup_{i=1}^{n_q} \overline{\Omega_i} \right) \quad \text{with} \quad \Omega_i \cap \Omega_j = \varnothing \quad \text{for} \quad i \neq j,$$

The shape functional and shape derivative are ($M = 1$)

$$\mathcal{J}(\{\Omega_i\}) = \int_\Sigma |u(\{\Omega_i\}) - m|^2 + \beta \sum_{i=1}^{n_q} |q - q_i| Per(\Gamma_i),$$

$$d\mathcal{J}(\{\Omega_i\}; V) = \sum_{i=1}^{n_q} \int_{\Gamma_i} \left[(q_i - q) \nabla p \cdot \nabla u + \beta |q - q_i| \mathcal{H} \right] v_n.$$

Step 3: Conductivity optimization

We set

$$J(q) = \frac{1}{2}\sum_{i=1}^{M} \|u_i(q) - m_i\|^2_{L^2(\Sigma)} + \alpha R(q).$$

with $M = 1$. Consider perturbations of the conductivity

$$q_i^\eta = q_i + \eta \bar{q}_i,$$

which leads to the derivative

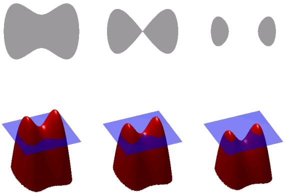
$$dJ(\{q_j\}_{j=1}^{n_q}; \bar{q}_i) = \bar{q}_i \left(\frac{q_i - q}{|q_i - q|} \beta Per(\Gamma_i) - \int_{\Omega_i} \nabla u \cdot \nabla p \right)$$

Level set methods

Choose a function $\phi(t, x)$ such that

$$\begin{aligned} \Omega_t &= \{x \in \Omega_t \mid \phi(t,x) < 0\} \\ \Omega_t^c &= \{x \in \Omega_t \mid \phi(t,x) > 0\} \\ \partial\Omega_t &= \{x \in \Omega_t \mid \phi(t,x) = 0\} \end{aligned}$$

For instance, ϕ can be chosen as the *signed distance function* to $\partial\Omega_t$



Evolution of the level set function

Consider a point $x(t)$ on the moving boundary Γ_t, we have $\phi(t, x(t)) = 0$. Differentiating w.r.t. t we get

$$\phi_t(t, x) + V(t, x) \cdot \nabla\phi(t, x) = 0.$$

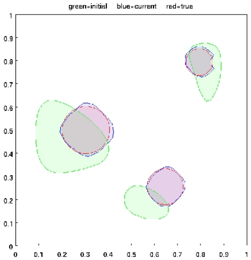
Since $\nabla\phi(t, x) = |\nabla\phi(t, x)| n(t, x)$
we get the Hamilton-Jacobi equation :

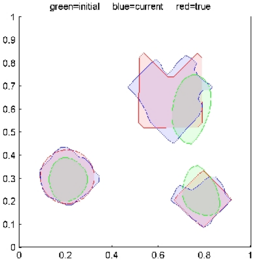
$$\phi_t(t, x) + v_n(t, x)|\nabla\phi(t, x)| = 0,$$

with ϕ_t time derivative of ϕ and $\phi(0, x)$ a given data.

Electrical Impedance Tomography

Reconstructions for 1% noise

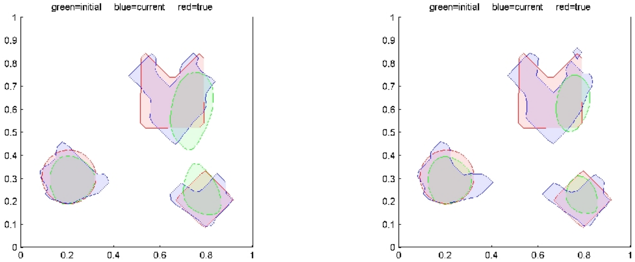




1. original phantom
2. reconstruction
3. topological derivative

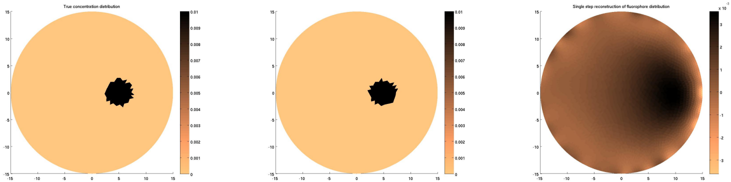
Electrical Impedance Tomography

Reconstructions for 3% (left) and 5% (right) noise.



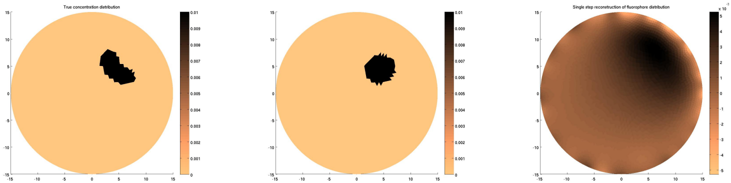
1. original phantom
2. reconstruction
3. topological derivative

Fluorescence Optical Tomography



1. original phantom (5% noise in the data)
2. reconstruction using topological derivative and exact c_1
3. reconstruction using single step algorithm [Egger et al.]

Fluorescence Optical Tomography



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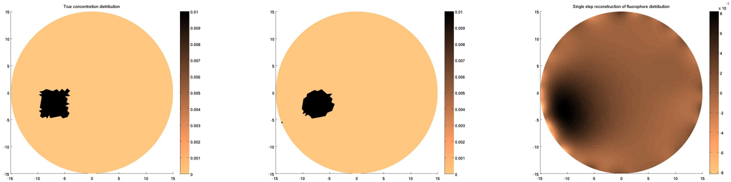
Fluorescence Optical Tomography



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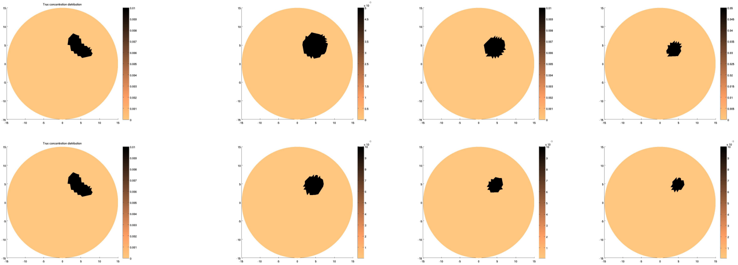
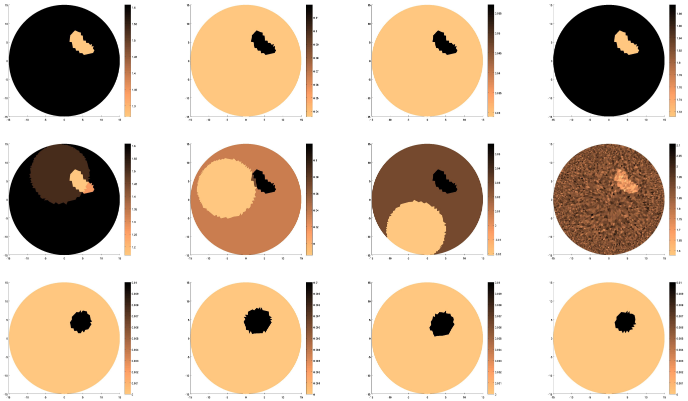
Fluorescence Optical Tomography



1. original phantom (left column)
2. reconstructed inclusion with trial values $c_0 = 0$ and $c_1 = 5.10^{-3}, 1.10^{-2}, 5.10^{-2}$ (first row),
3. reconstructed inclusion with trial values $c_0 = 3.10^{-5}, 7.10^{-5}, 1.10^{-4}$ and $c_1 = 1.10^{-2}$ (second row)

Fluorescence Optical Tomography



1. original coefficients $\tilde{\kappa}_x, \tilde{\mu}_x, \tilde{\kappa}_m, \tilde{\mu}_m$.
2. purposely erroneous coefficients $\kappa_x, \mu_x, \kappa_m, \mu_m$ used to compute the topological derivative.
3. corresponding reconstructions (third row)

Fluorescence Optical Tomography

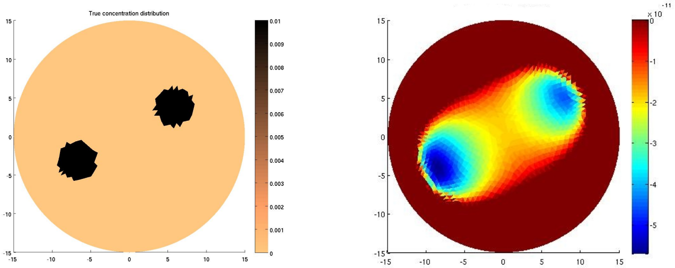


Figure : original phantom (left), topological derivative (right)

Inverse Potential Problem - Gravimetry

- Reconstruct an unknown measure with support in a domain Ω from a single measurement of its potential on the boundary $\partial\Omega$.
- Application to gravimetry: determine Earth's density distribution from the measurement of the gravity and its derivatives on the surface of the Earth.
- Ill-posed problem. A priori assumptions on the class of measures to be reconstructed can be made.
- Joint work with A. Canelas and A.A. Novotny.

Inverse Potential Problem - Mathematical model

$\Omega \subset \mathbb{R}^2$ open and bounded, with Lipschitz boundary $\partial\Omega$ and $\gamma = (\gamma_0, \gamma_1) \in \mathbb{R}^2$ is given.

$$PC_\gamma(\Omega) := \{b = \gamma_0 \chi_{\Omega\setminus\omega} + \gamma_1 \chi_\omega \in L^\infty(\Omega) \mid \omega \subset \Omega \text{ measurable}\}$$

Given $q^* \in H^{-1/2}(\partial\Omega)$ and $u^* \in H^{1/2}(\partial\Omega)$, find

$$b^* = \gamma_0 \chi_{\Omega\setminus\omega^*} + \gamma_1 \chi_{\omega^*} \in PC_\gamma(\Omega) \ ,$$

such that

$$\left\{ \begin{array}{rcl} -\Delta u & = & b^* \quad \text{in} \quad \Omega \ , \\ u & = & u^* \\ -\partial_n u & = & q^* \end{array} \right. \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{on} \quad \partial\Omega \ .$$

has a solution $u \in H^1(\Omega)$.

Inverse Potential Problem

Theorem (Isakov)

Assume $b_i = \gamma_0 \chi_{\Omega \setminus \omega_i} + \gamma_1 \chi_{\omega_i}$, $i = 1, 2$ where $\gamma = (\gamma_0, \gamma_1)$ is given, and ω_1, ω_2 are two star-shaped domains with respect to their barycenters. If the corresponding boundary data are equal, then $\omega_1 = \omega_2$.

We consider a broader class of admissible sets $\omega \subset PC_\gamma(\Omega)$:

$$\omega = \bigcup_{i \in \mathcal{I}} \omega_i \quad \text{with} \quad \omega_i \cap \omega_j = \emptyset \quad \text{for} \quad i \neq j \, .$$

with ω_i measurable and simply connected.

Inverse Potential Problem

Kohn-Vogelius functional

$$\min_{b \in PC_\gamma(\Omega)} J(b) := \frac{1}{2} \int_\Omega \left(u^D[b] - u^N[b] \right)^2 ,$$

where $u^D[b]$ and $u^N[b]$ solve (with $c[b] = \frac{1}{|\Omega|} \left(\int_{\partial\Omega} q^* - \int_\Omega b \right)$)

$$\begin{cases} -\Delta u^D[b] &= b \quad \text{in} \quad \Omega , \\ u^D[b] &= u^* \quad \text{on} \quad \partial\Omega , \end{cases}$$

$$\begin{cases} -\Delta u^N[b] &= b + c[b] \quad \text{in} \quad \Omega , \\ -\partial_n u^N[b] &= q^* \quad \text{on} \quad \partial\Omega , \\ \int_\Omega u^N[b] &= \int_\Omega u^D[b] , \end{cases}$$

Inverse Potential Problem

Define $\varpi_{\mathbf{e},\hat{\mathbf{x}}} = \cup_{i\in\mathcal{I}} B(\varepsilon_i, \widehat{x}_i)$ and

$$b_{\mathbf{e},\hat{\mathbf{x}}} = \gamma_0 \chi_{\Omega\setminus\varpi_{\mathbf{e},\hat{\mathbf{x}}}} + \gamma_1 \sum_{i\in\mathcal{I}} \chi_{B(\varepsilon_i,\widehat{x}_i)} \, .$$

We have the following expansion

$$\mathcal{J}(\Omega\setminus\varpi_{\mathbf{e},\hat{\mathbf{x}}}) = \mathcal{J}(\Omega) + \sum_{i\in\mathcal{I}} f_1(\varepsilon_i) D_T^1 \mathcal{J}(\widehat{x}_i) + \sum_{i,j\in\mathcal{I}} f_2(\varepsilon_i,\varepsilon_j) D_T^2 \mathcal{J}(\widehat{x}_i,\widehat{x}_j) \, ,$$

where $f_1(\varepsilon_i) = \pi\varepsilon_i^2$, $f_2(\varepsilon_i,\varepsilon_j) = \frac{1}{2}\pi^2\varepsilon_i^2\varepsilon_j^2$.

$$D_T^1 \mathcal{J}(\widehat{x}_i) = \int_\Omega (u^D[\gamma_0] - u^N[\gamma_0]) h_i \, , \quad D_T^2 \mathcal{J}(\widehat{x}_i,\widehat{x}_j) = \int_\Omega h_i h_j .$$

Inverse Potential Problem

Introduce the adjoint states

$$\begin{cases} -\Delta p^D = -(u^D[\gamma_0] - u^N[\gamma_0]) & \text{in} \quad \Omega\,, \\ p^D = 0 & \text{on} \quad \partial\Omega\,, \end{cases}$$

and

$$\begin{cases} -\Delta p^N = u^D[\gamma_0] - u^N[\gamma_0] & \text{in} \quad \Omega\,, \\ -\partial_n p^N = 0 & \text{on} \quad \partial\Omega\,, \\ \displaystyle\int_\Omega p^N = 0\,, \end{cases}$$

From Green's formula we get

$$D_T^1 \mathcal{J}(\widehat{x}_i) = -(\gamma_1 - \gamma_0)\left(p^D(\widehat{x}_i) + p^N(\widehat{x}_i)\right).$$

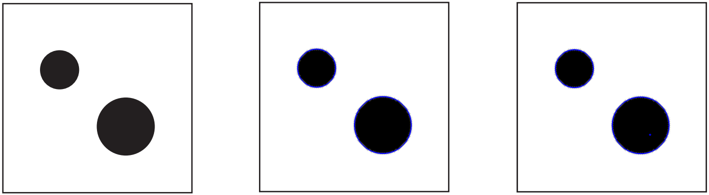
Inverse Potential Problem

- Define $a_i := \pi \varepsilon_i^2, i \in \mathcal{I}$,
- For fixed $\widehat{\mathbf{x}}$ minimize $J_{\widehat{\mathbf{x}}}(\mathbf{a}) := J(b_{\mathbf{e},\hat{\mathbf{x}}})$.
- To find $\mathbf{a}$ we differentiate the topological expansion to obtain the first order optimality conditions:

$$\sum_{j \in \mathcal{I}} D_T^2 \mathcal{J}(\widehat{x}_i, \widehat{x}_j) a_j = -D_T^1 \mathcal{J}(\widehat{x}_i) \quad \text{for} \quad i \in \mathcal{I} \,, \tag{1}$$

- Define $\mathbf{e}(\hat{\mathbf{x}}) := \sqrt{\mathbf{a}/\pi}$.
- Minimize $J(b_{\mathbf{e}(\hat{\mathbf{x}}),\hat{\mathbf{x}}})$ with respect to $\hat{\mathbf{x}}$.

Inverse Potential Problem



(a) (b) (c)

Figure : Looking for two anomalies: true source term (a) and reconstructions using two (b) and three trial balls (c).

Inverse Potential Problem

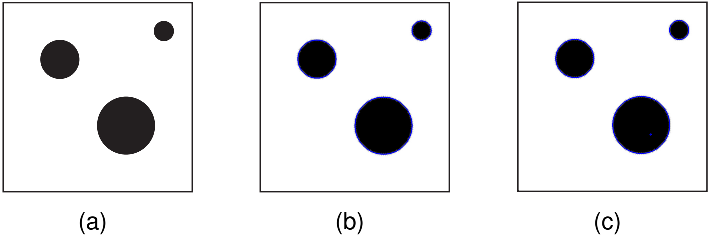
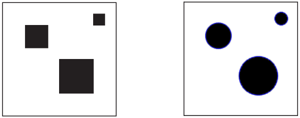


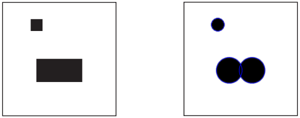
Figure : Looking for three anomalies: true source term (a) and reconstructions using three (b) and four trial balls (c).

Inverse Potential Problem



(a) (b)

Figure : Three anomalies: true source term (a) and reconstruction using three balls (b).



(a) (b)

Figure : Two anomalies: true source term (a) and reconstruction using three balls (b).

THANK YOU!